

# Geodesic disks in asymptotic Teichmüller space

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## Abstract

Let  $S$  be a hyperbolic Riemann surface. In a finite-dimensional Teichmüller space  $T(S)$ , it is still an open problem whether the geodesic disk passing through two points is unique. In an infinite-dimensional Teichmüller space it is also unclear how many geodesic disks pass through a Strebel point and the basepoint while we know that there are always geodesic disks passing through a non-Strebel point and the basepoint. In this paper, we answer the problem arising in the universal asymptotic Teichmüller space and prove that there are always infinitely many geodesic disks passing through two points.

## 1. Introduction

Let  $S$  be a hyperbolic Riemann surface, that is, it is covered by a holomorphic map:  $\varpi : \Delta \rightarrow S$ , where  $\Delta = \{|z| < 1\}$  is the open unit disk. Let  $T(S)$  be the Teichmüller space of  $S$ . A quotient space of the Teichmüller space  $T(S)$ , called the asymptotic Teichmüller space and denoted by  $AT(S)$ , was introduced by Gardiner and Sullivan (see [12] for  $S = \Delta$  and by Earle, Gardiner and Lakic for arbitrary hyperbolic  $S$  [2, 3, 11]).

$AT(S)$  is interesting only when  $T(S)$  is infinite dimensional, which occurs when  $S$  has border or when  $S$  has infinite topological type, otherwise,  $AT(S)$  consists of just one point. In recent years, the asymptotic space  $AT(S)$  and its tangent space are extensively studied, for examples, one can refer to [2, 3, 6, 9, 21, 22, 25].

We shall use some geometric terminologies adapted from [1] by Busemann. Let  $X$  and  $Y$  be metric spaces. An isometry of  $X$  into  $Y$  is a distance preserving map. A straight line in  $Y$  is a (necessarily closed) subset  $L$  that is an isometric image of the real line  $\mathbb{R}$ . A geodesic in  $Y$  is an isometric image of a non-trivial compact interval of  $\mathbb{R}$ . Its endpoints are the images of the endpoints of the interval, and we say that the geodesic joins its endpoints.

Geodesics play an important role in the theory of Teichmüller spaces. In an finite-dimensional Teichmüller space  $T(S)$ , there is always a unique geodesic connecting two

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points. The geometry is substantially different in an infinite dimensional Teichmüller space (see [4, 17, 18, 19, 24]). Generally, for a Strebel point, there is a unique geodesic connecting it and the basepoint. The situation on geodesics in the asymptotic Teichmüller space is still unclear while Fan [7] gave certain examples to show the nonuniqueness of geodesics in asymptotic spaces.

Another important role in the Teichmüller theory is the geodesic disk which is defined as the image of an isometric embedding from  $\Delta$  into the Teichmüller space  $T(S)$  with respect to the hyperbolic metric on  $\Delta$  and the Teichmüller metric on  $T(S)$  respectively. Using holomorphic motion, Earle et al. considered holomorphic geodesic disks containing two points in [5]. In [20], Li proved that there are always infinitely many geodesic disks passing through a non-Strebel point and the basepoint. So far, we know little information on how many geodesic disks passing through a Strebel point and the basepoint. It is even unknown whether the geodesic disk passing through two points is unique in a finite-dimensional Teichmüller space.

The motivation of the paper is to investigate geodesic disks in the asymptotic Teichmüller space. We characterize the nonuniqueness of geodesic disks in the universal asymptotic space  $AT(\Delta)$  completely. That is,

**Theorem 1.** *In the universal asymptotic Teichmüller space  $AT(\Delta)$ , there are always infinitely many geodesic disks containing two points.*

This paper is organized as follows. In Section 2, we introduce some basic notion in the Teichmüller space theory. In Section 3, an infinitesimal inequality of the asymptotic Teichmüller metric is founded. Theorem 1 is proved in Section 4. A parallel version of Theorem 1 in the infinitesimal setting is obtained in the last section.

The method used here can also be used to deal with some more general cases. However, there are some difficulties in solving the problem in all cases.

## 2. Some Preliminaries

### 2.1 Teichmüller space and asymptotic Teichmüller space

Let  $S$  be a Riemann surface of topological type. The Teichmüller space  $T(S)$  is the space of equivalence classes of quasiconformal maps  $f$  from  $S$  to a variable Riemann surface  $f(S)$ . Two quasiconformal maps  $f$  from  $S$  to  $f(S)$  and  $g$  from  $S$  to  $g(S)$  are equivalent if there is a conformal map  $c$  from  $f(S)$  onto  $g(S)$  and a homotopy through quasiconformal maps  $h_t$  mapping  $S$  onto  $g(S)$  such that  $h_0 = c \circ f$ ,  $h_1 = g$  and  $h_t(p) = c \circ f(p) = g(p)$  for every  $t \in [0, 1]$  and every  $p$  in the ideal boundary of  $S$ . Denote by  $[f]$  the Teichmüller equivalence class of  $f$ ; also sometimes denote the equivalence class by  $[\mu]$  where  $\mu$  is the Beltrami differential of  $f$ .

The asymptotic Teichmüller space is the space of a larger equivalence classes. The definition of the new equivalence classes is exactly the same as the previous definition with one exception; the word *conformal* is replaced by *asymptotically conformal*. A quasiconformal map  $f$  is asymptotically conformal if for every  $\epsilon > 0$ , there is a compact subset  $E$  of  $S$ , such that the dilatation of  $f$  outside of  $E$  is less than  $1 + \epsilon$ . Accordingly, denote by  $[[f]]$  or  $[[\mu]]$  the asymptotic equivalence class of  $f$ .

Denote by  $Bel(S)$  the Banach space of Beltrami differentials  $\mu = \mu(z)d\bar{z}/dz$  on  $S$  with finite  $L^\infty$ -norm and by  $M(S)$  the open unit ball in  $Bel(S)$ .

For  $\mu \in M(S)$ , define

$$k_0([\mu]) = \inf\{\|\nu\|_\infty : \nu \in [\mu]\}.$$

Define  $h^*(\mu)$  to be the infimum over all compact subsets  $E$  contained in  $S$  of the essential supremum norm of the Beltrami differential  $\mu(z)$  as  $z$  varies over  $S \setminus E$  and  $h([\mu])$  to be the infimum of  $h^*(\nu)$  taken over all representatives  $\nu$  of the class  $[\mu]$ . It is obvious that  $h([\mu]) \leq k_0([\mu])$ . Following [4],  $[\mu]$  is called a Strebel point if  $h([\mu]) < k_0(\tau)$ ; otherwise,  $\tau$  is called a non-Strebel point.

Put

$$h([\mu]) = \inf\{h^*(\nu) : \nu \in [\mu]\}.$$

We say that  $\mu$  is extremal in  $[\mu]$  if  $\|\mu\|_\infty = k_0([\mu])$  and  $\mu$  is asymptotically extremal if  $h^*(\mu) = h([\mu])$ . The relation  $h([\mu]) = h([\mu])$  is due to the definition.

The Teichmüller metric  $d_T$  between two points  $\tau, \sigma \in T(S)$  is defined as follows:

$$d_T(\tau, \sigma) = \frac{1}{2} \inf_{\mu \in \tau, \nu \in \sigma} \log \frac{1 + \|(\mu - \nu)/(1 - \bar{\nu}\mu)\|_\infty}{1 - \|(\mu - \nu)/(1 - \bar{\nu}\mu)\|_\infty}.$$

The asymptotic Teichmüller metric  $d_{AT}$  between two points  $\tilde{\tau}, \tilde{\sigma} \in AT(S)$  is defined by

$$d_{AT}(\tilde{\tau}, \tilde{\sigma}) = \frac{1}{2} \inf_{\mu \in \tilde{\tau}, \nu \in \tilde{\sigma}} \log \frac{1 + \|(\mu - \nu)/(1 - \bar{\nu}\mu)\|_\infty}{1 - \|(\mu - \nu)/(1 - \bar{\nu}\mu)\|_\infty}.$$

In particular, the distance between  $[[\mu]]$  and the basepoint  $[[0]]$  is

$$d_{AT}([[\mu]], [[0]]) = \frac{1}{2} \log H([\mu]), \text{ where } H([\mu]) = \frac{1 + h([\mu])}{1 - h([\mu])}.$$

## 2.2 Tangent spaces to Teichmüller space and asymptotic Teichmüller space

The cotangent space to  $T(S)$  at the basepoint is the Banach space  $Q(S)$  of integrable holomorphic quadratic differentials  $\varphi$  on  $S$  with  $L^1$ -norm

$$\|\varphi\| = \iint_S |\varphi(z)| dx dy < \infty.$$

In what follows, let  $Q^1(S)$  denote the unit sphere of  $Q(S)$ . Moreover, let  $Q_d^1(S)$  denote the set of all degenerating sequence  $\{\varphi_n\} \subset Q^1(S)$ . By definition, a sequence  $\{\varphi_n\}$  is called degenerating if it converges to 0 uniformly on compact subset of  $S$ .

Two Beltrami differentials  $\mu$  and  $\nu$  in  $Bel(S)$  are said to be infinitesimally equivalent if

$$\iint_S (\mu - \nu)\varphi dx dy = 0, \text{ for any } \varphi \in Q(S).$$

The tangent space  $Z(S)$  of  $T(S)$  at the basepoint is defined as the set of the quotient space of  $Bel(S)$  under the equivalence relations. Denote by  $[\mu]_Z$  the equivalence class of

$\mu$  in  $Z(S)$ . The set of all Beltrami differentials equivalent to zero is called the  $N$ -class in  $Bel(S)$ .

$Z(S)$  is a Banach space and actually [11] its standard sup-norm satisfies

$$\|[\mu]_Z\| := \sup_{\varphi \in Q^1(S)} \operatorname{Re} \iint_S \mu \varphi dx dy = \inf\{\|\nu\|_\infty : \nu \in [\mu]_Z\}.$$

Two Beltrami differentials  $\mu$  and  $\nu$  in  $Bel(S)$  are said to be infinitesimally asymptotically equivalent if

$$\sup_{Q_d^1(S)} \limsup_{n \rightarrow \infty} \operatorname{Re} \iint_S (\mu - \nu) \varphi_n dx dy = 0,$$

where the first *supremum* is taken when  $\{\varphi_n\}$  varies over  $Q_d^1(S)$ .

The tangent space  $AZ(S)$  of  $AT(S)$  at the basepoint is defined as the set of the quotient space of  $Bel(S)$  under the asymptotic equivalence relation. Denote by  $[[\mu]]_{AZ}$  the equivalence class of  $\mu$  in  $AZ(S)$ . The set of all Beltrami differentials equivalent to zero is called the  $Z_0$ -class in  $Bel(S)$ .

Define  $b([\mu]_Z)$  to be the infimum over all elements in the equivalence class  $[\mu]_Z$  of the quantity  $b^*(\nu)$ . Here  $b^*(\nu)$  is the infimum over all compact subsets  $E$  contained in  $S$  of the essential supremum of the the Beltrami differential  $\nu$  as  $z$  varies over  $S - E$ . It is obvious that  $b^*(\mu) \leq \|[\mu]_Z\|$ .  $[\mu]_Z$  is called an infinitesimal Strebel point if  $b([\mu]_Z) < \|[\mu]_Z\|$ . We say a Beltrami differential  $\mu \in Bel(S)$  vanishing at infinity if  $b^*(\mu) = 0$ .

Put

$$b([[ \mu ]]_{AZ}) = \inf\{b^*(\nu) : \nu \in [[ \mu ]]_{AZ}\}.$$

We say that  $\mu$  is (infinitesimally) extremal if  $\|\mu\|_\infty = \|[\mu]_Z\|$  and  $\mu$  is (infinitesimally) asymptotically extremal if  $b^*(\mu) = b([[ \mu ]]_{AZ})$ . We also have  $b([\mu]_Z) = b([[ \mu ]]_{AZ})$  [11].

$AZ(S)$  is a Banach space and its standard infinitesimal asymptotic norm satisfies (see [11])

$$\|[[ \mu ]]_{AZ}\| := \sup_{Q_d^1(S)} \limsup_{n \rightarrow \infty} \operatorname{Re} \iint_S \mu \varphi_n dx dy = \inf\{\|\nu\|_\infty : \nu \in [[ \mu ]]_{AZ}\} = b([[ \mu ]]_{AZ}).$$

### 2.3 Substantial boundary points and Hamilton sequence

Now we define the notion of boundary dilatation of a quasiconformal mapping at a boundary point. For a Riemann surface, the meaning of what is a boundary point can be problematic. However, if  $S$  can be embedded into a larger surface  $\tilde{S}$  such that the closure of  $S$  in  $\tilde{S}$  is compact, then it is possible to define the boundary dilatation. From now on, we assume that  $S$  is such a surface.

Let  $p$  be a point on  $\partial S$  and let  $\mu \in Bel(S)$ . Define

$$h_p^*(\mu) = \inf_{z \in U \cap S} \{\operatorname{esssup} |\mu(z)| : U \text{ is an open neighborhood in } \tilde{S} \text{ containing } p\}$$

to be the boundary dilatations of  $\mu$  at  $p$ . If  $\mu \in M(S)$ , define

$$h_p([\mu]) = \inf\{h_p^*(\nu) : \nu \in [\mu]\}$$

to be the boundary dilatations  $[\mu]$  at  $p$ . For a general  $\mu \in Bel(S)$ , define

$$b_p([\mu]_Z) = \inf\{h_p^*(\nu) : \nu \in [\mu]_Z\}$$

to be the boundary dilatations of  $[\mu]_Z$  at  $p$ . If we define the quantities

$$h_p([[ \mu ]]) = \inf\{h_p^*(\nu) : \nu \in [[ \mu ]]\}, \quad b_p([[ \mu ]])_{AZ} = \inf\{h_p^*(\nu) : \nu \in [[ \mu ]])_{AZ}\},$$

then  $h_p([\mu]) = h_p([[ \mu ]])$  and  $b_p([\mu]_Z) = b_p([[ \mu ]])_{AZ}$ . In particular, Lakic [15] proved that when  $S$  is a plane domain,

$$h([[ \mu ]]) = \max_{p \in \partial S} h_p([[ \mu ]]), \quad b([[ \mu ]])_{AZ} = \max_{p \in \partial S} b_p([[ \mu ]])_{AZ}.$$

As is well known,  $\mu$  is extremal if and only if it has a so-called Hamilton sequence, namely, a sequence  $\{\psi_n\} \subset Q^1(S)$ , such that

$$(2.1) \quad \lim_{n \rightarrow \infty} \operatorname{Re} \iint_S \mu \psi_n(z) dx dy = \|\mu\|_\infty.$$

Similarly, by Theorem 8 on page 281 in [11],  $\mu$  is asymptotically extremal if and only if it has an asymptotic Hamilton sequence, namely, a degenerating sequence  $\{\psi_n\} \subset Q^1(S)$ , such that

$$(2.2) \quad \lim_{n \rightarrow \infty} \operatorname{Re} \iint_S \mu \psi_n(z) dx dy = h^*(\mu).$$

Now, we assume that  $S$  is a plane domain with two or more boundary points. Then, the following lemma derives from Theorem 6 on page 333 in [11]:

**Lemma 2.1.** *The following three conditions are equivalent for every boundary point  $p$  of  $S$  and every asymptotic or infinitesimal asymptotic extremal representative  $\mu$ :*

- (1)  $h([\mu]) = h_p([\mu])$  (equivalently,  $h([[ \mu ]]) = h_p([[ \mu ]])$ ),
- (2)  $b([\mu]) = b_p([\mu])$  (equivalently,  $b([[ \mu ]])_{AZ} = b_p([[ \mu ]])_{AZ}$ ),
- (3) *there exists an asymptotic Hamilton sequence for  $\mu$  degenerating towards  $p$ , i.e., a sequence  $\{\psi_n\} \subset Q^1(S)$  converging uniformly to 0 on compact subsets of  $S \setminus \{p\}$ , such that*

$$(2.3) \quad \lim_{n \rightarrow \infty} \operatorname{Re} \iint_S \mu \psi_n(z) dx dy = h_p^*(\mu).$$

If one of three conditions in the lemma holds at some  $p \in \partial S$ , we call  $p$  is a substantial boundary point for  $[[ \mu ]]$  (or  $[\mu]$ ) and  $[[ \mu ]])_{AZ}$  (or  $[\mu]_Z$ ), respectively.

### 3. An infinitesimal inequality for asymptotic Teichmüller metric

**Theorem 2.** *Given  $\mu$  and  $\nu$  two Beltrami differentials in  $Bel(S)$ , then we have,*

$$(3.1) \quad \liminf_{t \rightarrow 0^+} \frac{d_{AT}([t\mu], [t\nu])}{t} \geq \sup_{Q_d^1(S)} \limsup_{n \rightarrow \infty} \operatorname{Re} \iint_S (\mu - \nu) \phi_n dx dy,$$

where the first supremum is taken when  $\{\phi_n\}$  varies over  $Q_d^1(S)$ .

We have an important corollary.

**Corollary 1.** *Let  $\mu$  and  $\nu$  be two asymptotically extremal Beltrami differentials in  $[[\mu]]$ . If the two geodesics  $[[t\mu]]$  and  $[[t\nu]]$  ( $0 \leq t \leq 1$ ) coincide, then*

$$\sup_{Q_d^1(S)} \limsup_{n \rightarrow \infty} \operatorname{Re} \iint_S (\mu - \nu) \phi_n dx dy = 0;$$

in other words,  $\mu$  and  $\nu$  are infinitesimally asymptotically equivalent.

The corollary is equivalent to Theorem 5.1 in [7].

To prove Theorem 2, we need the asymptotic fundamental inequality [3, 11], which is an asymptotic analogue of the well known Reich-Strebel inequality [23].

**The Asymptotic Fundamental Inequality.** Suppose  $f$  is a quasiconformal mapping from  $S$  to  $S^\mu$  with  $\mu$  its Beltrami differential. Let  $H = H([[\mu]])$ . Then

$$(3.2) \quad \frac{1}{H} \leq \liminf_{n \rightarrow \infty} \iint_S \frac{|1 - \mu \frac{\phi_n}{|\phi_n|}|^2}{1 - |\mu|^2} |\phi_n| dx dy,$$

for all degenerating sequences  $\{\phi_n\} \in Q_d^1(S)$ .

**Proof of Theorem 2.** Regard  $S$  as  $\Delta/\Gamma$ , where  $\Gamma$  is a Fuchsian group. Let  $\mu$  and  $\nu$  be two Beltrami differentials in  $Bel(S)$ . For each  $t > 0$  sufficiently close to zero, there exist two Riemann surfaces  $S_t$ ,  $R_t$  and two quasiconformal mappings  $f_t = f^{t\mu} : S \rightarrow R_t$ ,  $g_t = f^{t\nu} : S \rightarrow S_t$ , such that the Beltrami differentials of  $f_t$  and  $g_t$  are  $t\mu$  and  $t\nu$ , respectively. Suppose  $G_t : \Delta \rightarrow \Delta$  is the lift of  $g_t$  with the points 1,  $i$  and  $-1$  fixed. We can write  $S_t = \Delta/\Gamma_t$ , where

$$\Gamma_t = \{G_t \circ \gamma \circ G_t^{-1} | \gamma \in \Gamma\}$$

is a Fuchsian group.

Let  $\Omega$  be a fundamental domain of  $S$ . Then  $\Omega_t = G_t(\Omega)$  is a fundamental domain of  $S_t$ .

Let  $\varphi$  be an element of  $Q^1(S)$  and  $\tilde{\varphi}(z)dz^2$  be the lift of  $\varphi$ . Then  $\tilde{\varphi}$  satisfies

$$\tilde{\varphi}(\gamma(z))[\gamma'(z)]^2 = \tilde{\varphi}(z), \quad \gamma \in \Gamma, \quad z \in \Delta,$$

and

$$\iint_{\gamma(\Omega)} |\tilde{\varphi}| dx dy \equiv 1$$

for all  $\gamma \in \Gamma$ . There is a holomorphic quadratic differential  $\psi(z)dz^2 \in Q(\Delta)$  such that the Poincaré series of  $\psi$  (see [10], Chapter 4, Theorem 3)

$$(3.3) \quad \Theta\psi(z) = \sum_{\gamma \in \Gamma} \psi(\gamma(z))[\gamma'(z)]^2$$

is equal to  $\tilde{\varphi}$ . We define

$$\tilde{\phi}_t(z) = \sum_{\gamma_t \in \Gamma_t} \psi(\gamma_t(z))[\gamma'_t(z)]^2.$$

Putting

$$(3.4) \quad \tilde{\varphi}_t = \frac{\tilde{\phi}_t}{\iint_{\Omega_t} |\tilde{\phi}_t| dx dy},$$

we have

$$\tilde{\varphi}_t(\gamma_t(z))[\gamma'_t(z)]^2 = \tilde{\varphi}_t(z), \quad z \in \Delta,$$

and

$$(3.5) \quad \iint_{\gamma_t(\Omega_t)} |\tilde{\varphi}_t| dx dy \equiv 1$$

for all  $\gamma_t \in \Gamma_t$ , respectively. This means that  $\tilde{\varphi}_t(z)dz^2$  is a lift of a holomorphic differential  $\varphi_t \in Q^1(S_t)$ .

Let  $\Lambda_t$  be the composition of  $f_t$  and  $g_t^{-1}$ , i.e.,  $\Lambda_t = f_t \circ g_t^{-1} : S_t \rightarrow R_t$ . Denote by  $\lambda_t$  the complex dilatation of  $\Lambda_t$ . Let  $\tilde{\mu}$ ,  $\tilde{\nu}$  and  $\tilde{\lambda}_t$  be the lifts of  $\mu$ ,  $\nu$  and  $\lambda_t$ , respectively. Then we have

$$\tilde{\lambda}_t(w) = \left[ \frac{t(\tilde{\mu} - \tilde{\nu})}{1 - t^2 \tilde{\mu} \tilde{\nu}} \cdot \frac{\partial_z G_t}{\partial_z \overline{G_t}} \right] \circ G_t^{-1}(w).$$

Let  $H(t) = H([\lambda_t])$ . Put

$$h(t) = \frac{H(t) - 1}{H(t) + 1}.$$

We can find a degenerating sequence  $\{\varphi^n\}$  in  $Q_d^1(S)$ , such that

$$(3.6) \quad \lim_{n \rightarrow \infty} \operatorname{Re} \iint_S (\mu - \nu) \varphi^n dx dy = \sup_{Q_d^1(S)} \limsup_{n \rightarrow \infty} \operatorname{Re} \iint_S (\mu - \nu) \phi_n dx dy.$$

Then every  $\varphi^n$  corresponds to a  $\psi_n \in Q(\Delta)$ . Let  $\tilde{\varphi}_t^n$  be given by (3.4). The sequence  $\{\tilde{\varphi}_t^n\}$  is a lift of a sequence  $\{\varphi_t^n\}$  in  $Q^1(S_t)$  which is degenerating on  $S_t$ .

The Asymptotic Fundamental Inequality (3.2) implies

$$\frac{1}{H(t)} = \frac{1 - h(t)}{1 + h(t)} \leq \liminf_{n \rightarrow \infty} \iint_{S_t} \frac{\left| 1 - \lambda_t \frac{\varphi_t^n}{|\varphi_t^n|} \right|^2}{1 - |\lambda_t|^2} |\varphi_t^n| du dv,$$

which yields

$$h(t) \geq t \limsup_{n \rightarrow \infty} \operatorname{Re} \iint_{\Omega_t} \left[ (\tilde{\mu} - \tilde{\nu}) \cdot \frac{\partial_z G_t}{\overline{\partial_z G_t}} \right] \circ G_t^{-1}(w) \cdot \tilde{\varphi}_t^n(w) dudv + O(t^2),$$

equivalently,

$$h(t) \geq t \limsup_{n \rightarrow \infty} \operatorname{Re} \iint_{\Omega} (\tilde{\mu} - \tilde{\nu}) \cdot (\partial_z G_t)^2 |1 - t\tilde{\nu}|^2 \cdot \tilde{\varphi}_t^n(G_t) dx dy + O(t^2).$$

Therefore,

$$\frac{h(t)}{t} \geq \limsup_{n \rightarrow \infty} \operatorname{Re} \iint_{\Omega} (\tilde{\mu} - \tilde{\nu}) \cdot (\partial_z G_t)^2 \cdot \tilde{\varphi}_t^n(G_t) dx dy + O(t).$$

Furthermore, we have

$$\begin{aligned} \liminf_{t \rightarrow 0} \frac{h(t)}{t} &\geq \liminf_{t \rightarrow 0} \limsup_{n \rightarrow \infty} \operatorname{Re} \iint_{\Omega} (\tilde{\mu} - \tilde{\nu}) \cdot (\partial_z G_t)^2 \cdot \tilde{\varphi}_t^n(G_t) dx dy \\ &\geq \limsup_{n \rightarrow \infty} \liminf_{t \rightarrow 0} \operatorname{Re} \iint_{\Omega} (\tilde{\mu} - \tilde{\nu}) \cdot (\partial_z G_t)^2 \cdot \tilde{\varphi}_t^n(G_t) dx dy. \end{aligned}$$

We now show that for every  $n$ ,

$$(3.7) \quad \liminf_{t \rightarrow 0} \operatorname{Re} \iint_{\Omega} (\tilde{\mu} - \tilde{\nu}) \cdot (\partial_z G_t)^2 \cdot \tilde{\varphi}_t^n(G_t) dx dy = \operatorname{Re} \iint_{\Omega} (\tilde{\mu} - \tilde{\nu}) \tilde{\varphi}^n(z) dx dy.$$

We may assume that there is a sequence  $\{t_m : t_m \rightarrow 0\}$  such that

$$\begin{aligned} &\liminf_{t \rightarrow 0} \operatorname{Re} \iint_{\Omega} (\tilde{\mu} - \tilde{\nu}) \cdot (\partial_z G_t)^2 \cdot \tilde{\varphi}_t^n(G_t) dx dy \\ &= \lim_{m \rightarrow \infty} \operatorname{Re} \iint_{\Omega} (\tilde{\mu} - \tilde{\nu}) \cdot (\partial_z G_{t_m})^2 \cdot \tilde{\varphi}_{t_m}^n(G_{t_m}) dx dy. \end{aligned}$$

The fact that the Beltrami differential of  $G_t$  is  $t\tilde{\nu}$  implies that  $G_t$  is a good approximation of the identity map  $id$  on  $\Delta$ , and hence there exists a subsequence of  $\{t_m\}$ , say, itself, such that  $G_{t_m}$  converges to  $id$  uniformly on  $\overline{\Delta}$  and  $\partial_z G_{t_m}(z) \rightarrow 1$  almost everywhere in  $\Delta$  (see [16]). Thus,  $(\partial_z G_{t_m})^2 \rightarrow 1$  a.e. in  $\Delta$ . The fundamental domain  $\Omega_{t_m} = G_{t_m}(\Omega)$  converges to  $\Omega$ .  $\tilde{\varphi}_{t_m}^n$  converges to  $\tilde{\varphi}^n$  uniformly on compact subset of  $\Delta$  passing to a subsequence if necessary. (3.7) follows readily. Thus,

$$\liminf_{t \rightarrow 0} \frac{h(t)}{t} \geq \lim_{n \rightarrow \infty} \operatorname{Re} \iint_S (\mu - \nu) \varphi^n dx dy,$$

In terms of (3.6), (3.1) derives immediately. Theorem 2 is proved.



## 4. Geodesic disks in the universal asymptotic Teichmüller space

$[[\mu]]$  (or  $[[\mu]]_{AZ}$ ) is called a substantial point in  $AT(\Delta)$  (or  $AZ(\Delta)$ ), if every  $p \in \partial\Delta$  is a(n) (infinitesimal) substantial boundary point for  $[[\mu]]$  (or  $[[\mu]]_{AZ}$ ); otherwise,  $[[\mu]]$  (or  $[[\mu]]_{AZ}$ ) is called a non-substantial point.

Let  $SP$  and  $ISP$  denote the collection of all (infinitesimal) substantial points in  $AT(\Delta)$  and  $AZ(\Delta)$ , respectively. It is clear that  $AT(\Delta) \setminus SP$  and  $AZ(\Delta) \setminus ISP$  are open and dense in  $AT(\Delta)$  and  $AZ(\Delta)$ , respectively.

Let  $d_H(z_1, z_2)$  denote the hyperbolic distance between two points  $z_1, z_2$  in  $\Delta$ , i.e.,

$$d_H(z_1, z_2) = \frac{1}{2} \log \frac{1 + \left| \frac{z_1 - z_2}{1 - \bar{z}_1 z_2} \right|}{1 - \left| \frac{z_1 - z_2}{1 - \bar{z}_1 z_2} \right|}.$$

Recall that a geodesic disk in  $AT(S)$  is the image of a map  $\Psi : \Delta \hookrightarrow AT(S)$  which is an isometric embedding with respect to the hyperbolic metric on  $\Delta$  and the asymptotic Teichmüller metric on  $AT(S)$ , respectively.

We say that  $\mu$  is a non-Strebel extremal if it is an extremal representative in the non-Strebel point  $[\mu]$  (or  $[\mu]_Z$ ). Let  $\mu$  be a non-Strebel extremal with  $\|\mu\|_\infty = k \in (0, 1)$ . Then the embedding

$$\begin{aligned} \Psi_\mu : \Delta &\hookrightarrow AT(S), \\ t &\longmapsto [[t\mu/k]], \end{aligned}$$

is a holomorphic isometry. Because the subsequent Lemma 4.1 indicates that a non-Strebel extremal representative in  $[[\mu]]$  always exists, there is at least a geodesic disk containing  $[0]$  and  $[[\mu]]$ .

To obtain Theorem 1, we need a series of lemmas.

**Lemma 4.1.** *Let  $\mu \in Bel(S)$ . Then,*

- (1) *if  $\mu \in M(S)$ , then there exists a Beltrami differential  $\nu \in [[\mu]]$  such that  $\nu$  is a non-Strebel extremal;*
- (2) *there exists a Beltrami differential  $\nu \in [[\mu]]_{AZ}$  such that  $\nu$  is a non-Strebel extremal.*

*Proof.* We only show the first part (1).

By Theorem 2 on page 296 of [11], there is an asymptotic extremal representative in  $[[\mu]]$ , say  $\mu$ , such that  $h([\mu]) = h^*(\mu)$ . If  $h^*(\mu) = 0$ , let  $\nu$  be identically zero. If  $h^*(\mu) > 0$ , put

$$\nu(z) = \begin{cases} \mu(z), & |\mu(z)| \leq h^*(\mu), \\ h^*(\mu)\mu(z)/|\mu(z)|, & |\mu(z)| > h^*(\mu). \end{cases}$$

In either case, it is easy to verify that  $\nu \in [[\mu]]$  and is a non-Strebel extremal.  $\square$

**Lemma 4.2.** *Let  $\mu \in Bel(\Delta)$  and  $p \in \partial\Delta$ . Then,*

- (1) *if  $\mu \in M(\Delta)$ , then there exists a Beltrami differential  $\nu \in [[\mu]]$  such that  $\nu$  is a non-Strebel extremal and  $h_p^*(\nu) = h_p^*([\mu])$ ;*
- (2) *there exists a Beltrami differential  $\nu \in [[\mu]]_{AZ}$  such that  $\nu$  is a non-Strebel extremal and  $b_p^*(\nu) = b_p^*([\mu]_{AZ})$ .*

*Proof.* We also only show the first part (1).

Case 1.  $h_p([\mu]) = h([\mu]) := h$ .

The case has a trivial proof since any non-Strebel extremal representative in  $[[\mu]]$  has the required property.

Case 2.  $h_p([\mu]) < h([\mu]) := h$ .

By the definition of boundary dilatation, there exists a Beltrami differential  $\chi \in [[\mu]]$  such that  $h_p^*(\chi) < \min\{h_p([\mu]) + \frac{1}{2}, h\}$ .

Let  $E_n = \{z \in \Delta : |z - p| < \frac{1}{n}\}$ . Then  $E_n$  converges to  $\emptyset$  as  $n \rightarrow \infty$ .

There is a number  $n_1 \geq 1$  such that  $\|\chi\|_\infty < \min\{h_p([\mu]) + \frac{1}{2}, h\}$  restricted on  $E_{n_1}$ .

Restrict  $\chi$  on  $\Delta \setminus E_{n_1+1}$  and regard  $[[\chi]]$  as a point in  $AT(\Delta \setminus E_{n_1+1})$ . Then  $h([\chi]) = h$ . By Lemma 4.1, we can choose a non-Strebel extremal in  $[[\chi]]$ , say  $\chi_1(z)$ . Define

$$\nu_1(z) = \begin{cases} \chi_1(z), & z \in \Delta \setminus E_{n_1+1}, \\ \chi(z), & z \in E_{n_1+1}. \end{cases}$$

Then,  $\nu_1$  is a non-Strebel extremal in  $[[\mu]]$  and  $|\nu_1(z)| \leq \min\{h_p([\mu]) + \frac{1}{2}, h\}$  in  $E_{n_1+1}$  almost everywhere.

Consider  $\nu_1(z)$  on  $E_{n_1+1}$ . We still have  $h_p^*(\nu_1) = h_p([\mu])$ . By the same reason, there is a number  $n_2 > n_1 + 1$ , and a Beltrami differential  $\nu_2(z) \in M(E_{n_1+1})$  such that  $\nu_2 \in [[\nu_1]]$  regarded as a point in  $AT(E_{n_1+1})$ ,  $\|\nu_2\|_\infty < \min\{h_p([\mu]) + \frac{1}{2}, h\}$  and  $|\nu_2(z)| < \min\{h_p([\mu]) + \frac{1}{2^2}, h\}$  in  $E_{n_2+1}$  almost everywhere.

Following the construction, we get a sequence  $n_{j+1}$  ( $j \geq 1$ ),  $n_{j+1} \rightarrow \infty$  ( $j \rightarrow \infty$ ) and a sequence  $\nu_{j+1}(z) \in M(E_{n_j+1})$  such that  $\nu_{j+1} \in [[\nu_j]]$  regarded as a point in  $AT(E_{n_j+1})$ ,  $\|\nu_{j+1}\|_\infty < \min\{h_p([\mu]) + \frac{1}{2^j}, h\}$  and  $|\nu_{j+1}(z)| < \min\{h_p([\mu]) + \frac{1}{2^{j+1}}, h\}$  in  $E_{n_{j+1}+1}$  almost everywhere.

Define

$$\nu(z) = \begin{cases} \nu_1(z), & z \in \Delta \setminus E_{n_1+1}, \\ \nu_2(z), & z \in E_{n_1+1} \setminus E_{n_2+1}, \\ \vdots \\ \nu_j(z), & z \in E_{n_{j-1}+1} \setminus E_{n_j+1}, \\ \vdots \end{cases}$$

Then  $\nu$  belongs to  $[[\mu]]$  and is the desired non-Strebel extremal.  $\square$

**Lemma 4.3.** Let  $t_1$  and  $t_2$  be two complex numbers and  $k_1$  and  $k_2$  be two real numbers. Then we have

$$(4.1) \quad \left| \frac{(t_1 - t_2)k_1}{1 - \overline{t_2}t_1k_1^2} \right| \leq \left| \frac{(t_1 - t_2)k_2}{1 - \overline{t_2}t_1k_2^2} \right|, \text{ if } 0 < k_1 \leq k_2 \text{ and } k_2^2|t_1t_2| < 1.$$

*Proof.* Without any loss of generality, we may assume that  $t_1t_2 \neq 0$ . Let  $k$  be a real variable and put

$$F(k) = \left| \frac{(t_1 - t_2)k}{1 - \overline{t_2}t_1k^2} \right|^2 = \frac{|t_1 - t_2|^2k^2}{1 + |t_1t_2|^2k^4 - 2k^2\operatorname{Re}(\overline{t_2}t_1)}.$$

It is easy to verify that  $F'(k) \geq 0$  as  $k \in (0, 1/\sqrt{|t_1 t_2|})$ . Therefore  $F(k)$  is an increasing function on  $(0, 1/\sqrt{|t_1 t_2|})$  and hence (4. 1) holds.  $\square$

**Lemma 4.4.** *Given  $k \in (0, 1]$  and  $s, t \in \Delta$ , then*

$$(4. 2) \quad \left| \frac{(Re(s) - Re(t))k}{1 - Re(s)\overline{Re(t)}k^2} \right| \leq \left| \frac{t - s}{1 - s\bar{t}} \right|.$$

*Proof.* By Lemma 4.3, we have

$$\left| \frac{(Re(s) - Re(t))k}{1 - Re(s)\overline{Re(t)}k^2} \right| \leq \left| \frac{Re(s) - Re(t)}{1 - Re(s)\overline{Re(t)}} \right|.$$

On the other hand, Lemma 6.4 on page 75 of [13] indicates that

$$d_H(Re(t), Re(s)) = d_H\left(\frac{t + \bar{t}}{2}, \frac{s + \bar{s}}{2}\right) \leq d_H(t, s),$$

which implies that

$$\left| \frac{Re(s) - Re(t)}{1 - Re(s)\overline{Re(t)}} \right| \leq \left| \frac{t - s}{1 - s\bar{t}} \right|.$$

$\square$

**Proof of Theorem 1.** It suffices to prove that for any  $[[\mu]] (\neq [[0]])$  in  $AT(\Delta)$ , there are infinitely many geodesic disks passing through  $[[\mu]]$  and  $[[0]]$ . Choose a non-Strebel extremal representative in  $[[\mu]]$ , say  $\mu$ . Then  $k_0([\mu]) = h([\mu]) = h^*(\mu) := h$ .

*Case 1.*  $[[\mu]]$  is not a substantial point.

There is a point  $q \in \partial\Delta$  which is not a substantial boundary point for  $[[\mu]]$ . By Lemma 4.2, we may assume  $h_q^*(\mu) < h$  in addition.

By the definition of boundary dilatation, we can find a small neighborhood  $B(q)$  of  $q$  in  $\Delta$  such that  $|\mu(z)| \leq \rho < h$  for some  $\rho > 0$  in  $B(q)$  almost everywhere. Therefore for any  $\zeta \in \partial\Delta \cap \partial B(q)$ ,  $h_\zeta^*(\mu) \leq \rho$ .

Choose  $\delta(z) \in M(\Delta)$  such that  $\|\delta\|_\infty \leq \beta < h - \rho$  and  $\delta(z) = 0$  when  $z \in \Delta \setminus B(q)$ .

Let  $\Sigma$  be the collection of the complex-valued functions  $\sigma$  defined on  $\Delta$  with the following conditions:

- (A)  $\sigma$  is continuous with  $\sigma(0) = 0$  and  $\sigma(h) = 0$ ,
- (B)  $\left| \frac{|(s-t)\mu(z)/h| + |\sigma(s) - \sigma(t)|\beta}{1 - [s\mu(z)/h + \sigma(s)\delta(z)][t\mu(z)/h + \sigma(t)\delta(z)]} \right| \leq \left| \frac{s-t}{1-s\bar{t}} \right|, \quad t, s \in \Delta, z \in B(q).$

We claim that  $\Sigma$  contains uncountably many elements. At first, let  $\sigma$  be a Lipschitz continuous function on  $\Delta$  with the following conditions,

- (i) for some small  $\alpha > 0$ ,  $|\sigma(s) - \sigma(t)| < \alpha|s - t|$ ,  $t, s \in \Delta$ ,
- (ii)  $\sigma(0) = 0$  and  $\sigma(h) = 0$ ,
- (iii) for some small  $t_0$  in  $(0, h)$ ,  $\sigma(t) \equiv 0$  when  $|t| \geq t_0$ ,

Secondly, we show that when  $t_0$  and  $\alpha$  are sufficiently small,  $\sigma$  belongs to  $\Sigma$ , for which it suffices to show that  $\sigma$  satisfies the condition (B). Let  $t, s \in \Delta$ . It is no harm to assume that  $|t| \leq |s|$ .

Case 1.  $|t| \geq t_0$ .

Since  $\sigma(s) = \sigma(t) = 0$ , by Lemma 4.3, we have

$$\begin{aligned} & \left| \frac{|(s-t)\mu(z)/h| + |\sigma(t) - \sigma(s)|\beta}{1 - [s\mu(z)/h + \sigma(s)\delta(z)][\overline{t\mu(z)/h + \sigma(t)\delta(z)}]} \right| = \left| \frac{(s-t)\mu(z)/h}{1 - [s\mu(z)/h][\overline{t\mu(z)/h}]} \right| \\ & \leq \left| \frac{s-t}{1-\overline{st}} \right|, \quad z \in B(q). \end{aligned}$$

Case 2.  $|t| < t_0$ .

Put  $\gamma = \rho/h + \alpha\beta$  and choose small  $\alpha > 0$  such that  $\gamma < 1$ . On the one hand, since  $|\sigma(t)| \leq \alpha|t|$  and  $|\sigma(s)| \leq \alpha|s|$ , it holds that

$$\begin{aligned} & \left| \frac{|(s-t)\mu(z)/h| + |\sigma(t) - \sigma(s)|\beta}{1 - [s\mu(z)/h + \sigma(s)\delta(z)][\overline{t\mu(z)/h + \sigma(t)\delta(z)}]} \right| \leq \left| \frac{(s-t)(\rho/h + \alpha\beta)}{1 - [|s|\rho/h + \alpha|s|\beta][\overline{|t|\rho/h + \alpha|t|\beta}]} \right| \\ & \leq \left| \frac{(s-t)(\rho/h + \alpha\beta)}{1 - [\rho/h + \alpha\beta][\overline{t_0(\rho/h + \alpha\beta)}]} \right| = \gamma \left| \frac{s-t}{1-t_0\gamma^2} \right|, \quad z \in B(q). \end{aligned}$$

On the other hand, we have

$$\left| \frac{s-t}{1-\overline{st}} \right| \geq \left| \frac{s-t}{1+t_0} \right|.$$

When  $t_0$  is sufficiently small, we can get

$$\left| \frac{s-t}{1+t_0} \right| \geq \gamma \left| \frac{s-t}{1-t_0\gamma^2} \right|.$$

Therefore, when  $t_0$  and  $\alpha$  are sufficiently small,  $\sigma$  satisfies the condition (B).

For a given  $\sigma \in \Sigma$ , define for  $t \in \Delta$ ,

$$(4.3) \quad \mu_t(z) = \begin{cases} t\mu(z)/h, & z \in \Delta \setminus B(q), \\ t\mu(z)/h + \sigma(t)\delta(z), & z \in B(q), \end{cases}$$

and the map

$$\begin{aligned} \Psi_\sigma : \Delta &\hookrightarrow AT(\Delta), \\ t &\longmapsto [[\mu_t]]. \end{aligned}$$

It is obvious that  $\Psi_\sigma(\Delta)$  contains  $[[\mu]]$  and the basepoint  $[[0]]$ . We show that  $\Psi_\sigma$  is an isometric embedding. It is sufficient to verify that

$$(4.4) \quad d_{AT}([\mu_t], [\mu_s]) = d_H(t, s), \quad t, s \in \Delta,$$

where  $d_H$  expresses the hyperbolic distance on the unit disk.

Let  $f_s : \Delta \rightarrow \Delta$  and  $f_t : \Delta \rightarrow \Delta$  be quasiconformal mappings with Beltrami differentials  $\mu_s$  and  $\mu_t$  respectively. It is convenient to assume that  $t \neq 0$  and  $s \neq t$ . Set  $F_{s,t} = f_s \circ f_t^{-1}$  and assume that the Beltrami differential of  $F_{s,t}$  is  $\nu_{s,t}$ . Then a simple computation shows,

$$\nu_{s,t} \circ f_t(z) = \frac{1}{\tau} \frac{\mu_s(z) - \mu_t(z)}{1 - \overline{\mu_t(z)}\mu_s(z)},$$

where  $z = f_t^{-1}(w)$  for  $w \in \Delta$  and  $\tau = \overline{\partial f_t}/\partial f_t$ . We have

$$(4.5) \quad \nu_{s,t} \circ f_t(z) = \begin{cases} \frac{1}{\tau} \frac{s-t}{1-st|\mu(z)|^2/h^2} \frac{\mu(z)}{h}, & z \in \Delta \setminus B(q), \\ \frac{1}{\tau} \frac{(s-t)\mu(z)/h + [\sigma(s)-\sigma(t)]\delta(z)}{1-[s\mu(z)/h + \sigma(s)\delta(z)]t\mu(z)/h + \sigma(t)\delta(z)}, & z \in B(q). \end{cases}$$

Since  $\sigma \in \Sigma$ , due to condition (B) we see that restricted on  $f_t(B(q))$ ,

$$(4.6) \quad \|\nu_{s,t}\|_\infty \leq \left| \frac{s-t}{1-st} \right|.$$

Suppose  $p \in \partial\Delta$  is a substantial boundary point for  $[[\mu]]$ . By Lemma 2.1 there is a degenerating Hamilton sequence  $\{\psi_n\} \subset Q^1(\Delta)$  towards  $p$  such that

$$h = \lim_{n \rightarrow \infty} \operatorname{Re} \iint_{\Delta} \mu(z) \psi_n(z) dx dy.$$

Then, we have

$$|t| = \lim_{n \rightarrow \infty} \operatorname{Re} \iint_{\Delta} \mu_t(z) e^{-i \arg t} \psi_n(z) dx dy.$$

On the other hand, it is easy to see that  $h([[ \mu_t ]]) = h^*(\mu_t) = |t|$  and hence  $\mu_t$  is an asymptotic extremal. Therefore, the Beltrami differential  $\tilde{\mu}_t$  of  $f_t^{-1}$  is also an asymptotic extremal where  $\tilde{\mu}_t = -\mu_t(f_t^{-1}) \overline{\partial f_t^{-1}} / \partial f_t^{-1}$ .  $f_t(p)$  is a substantial boundary point for  $[[\tilde{\mu}_t]]$  and there is a degenerating Hamilton sequence  $\{\tilde{\psi}_n\} \subset Q^1(\Delta)$  towards  $f_t(p)$  such that

$$\lim_{n \rightarrow \infty} \operatorname{Re} \iint_{\Delta} \tilde{\mu}_t \tilde{\psi}_n(w) du dv = h([[ \tilde{\mu}_t ]]) = |t|.$$

Furthermore,

$$(4.7) \quad \begin{aligned} & \lim_{n \rightarrow \infty} \operatorname{Re} \iint_{\Delta} \nu_{s,t}(w) e^{i(\theta + \arg t)} \tilde{\psi}_n(w) du dv \\ &= \lim_{n \rightarrow \infty} \operatorname{Re} \iint_{\Delta} \frac{s-t}{1-st} \frac{\tilde{\mu}_t}{t} e^{i(\theta + \arg t)} \tilde{\psi}_n(w) du dv = \left| \frac{s-t}{1-st} \right|, \end{aligned}$$

where  $\theta = -\arg \frac{s-t}{1-st}$ .

Thus, by (4.6), (4.7) and Lemma 2.1, it follows that  $h([[ \nu_{s,t} ]]) = \left| \frac{s-t}{1-st} \right|$ ,  $\nu_{s,t}$  is asymptotically extremal and the equality (4.4) holds.

It remains to show that there are infinitely many geodesic disks passing through  $[[\mu]]$  and  $[[0]]$  when  $\sigma$  varies over  $\Sigma$  and  $\delta(z)$  varies over  $M(\Delta)$  suitably, respectively.

Firstly, choose  $\delta(z)$  in  $M(\Delta)$  such that

$$(4.8) \quad \sup_{Q_d^1(\Delta)} \limsup_{n \rightarrow \infty} \operatorname{Re} \iint_{\Delta} \delta \varphi_n dx dy = c > 0,$$

where the *supremum* is over all sequences  $\{\varphi_n\}$  in  $Q_d^1(\Delta)$  degenerating towards  $q$ .

Secondly, we choose small  $t_0$  in  $(0, h)$ , small  $\alpha > 0$  and  $\sigma \in \Sigma$  such that  $\sigma(t) \equiv 0$  whenever  $|t| \geq t_0$  and  $\sigma(t) = \alpha t$  when  $t \in [0, t_0/2]$ .

*Claim.* When  $\alpha$  varies in a small range, the geodesic disks  $\Psi_\sigma(\Delta)$  are mutually different.

Let  $\alpha_1$  and  $\alpha_2$  be two small different positive numbers and  $\sigma_j(t) = \alpha_j t$  when  $t \in [0, t_0]$  ( $j = 1, 2$ ), respectively. Now, the corresponding expression of equation (4. 3) is

$$\mu_t^j(z) = \begin{cases} t\mu(z)/h, & z \in \Delta \setminus B(q), \\ t\mu(z)/h + \sigma_j(t)\delta(z), & z \in B(q), \quad j = 1, 2. \end{cases}$$

When  $t \in [0, h]$ , they correspond to geodesics  $G_j = \{[\mu_t^j] : t \in [0, h]\}$  ( $j = 1, 2$ ), respectively. Note that when  $t \in [0, t_0/2]$ ,

$$\mu_t^j(z) = \begin{cases} t\mu(z)/h, & z \in \Delta \setminus B(q), \\ t\mu(z)/h + t\alpha_j\delta(z), & z \in B(q), \quad j = 1, 2. \end{cases}$$

Define

$$\mu^j(z) = \begin{cases} \mu(z)/h, & z \in \Delta \setminus B(q), \\ \mu(z)/h + \alpha_j\delta(z), & z \in B(q), \quad j = 1, 2. \end{cases}$$

Since

$$\begin{aligned} \sup_{Q_d^1(\Delta)} \limsup_{n \rightarrow \infty} Re \iint_{\Delta} (\mu^1 - \mu^2) \varphi_n dx dy &= \sup_{Q_d^1(\Delta)} \limsup_{n \rightarrow \infty} Re \iint_{\Delta} (\alpha_1 - \alpha_2) \delta \varphi_n dx dy \\ &\geq |\alpha_1 - \alpha_2| c > 0, \end{aligned}$$

by Theorem 2, the geodesics  $G_1$  and  $G_2$  are obviously different and hence the geodesic disks  $\Psi_{\sigma_1}(\Delta)$  and  $\Psi_{\sigma_2}(\Delta)$  are different.

If fix small  $\alpha > 0$  and let  $\delta$  vary suitably in  $M(\Delta)$ , then we can also get infinitely many geodesic disks as desired.

*Case 2.*  $[[\mu]]$  is a substantial point.

Fix a boundary point  $p \in \partial\Delta$ . Let  $B(p) = \{z \in \Delta : |z - p| < r\}$  for small  $r > 0$  and  $E = \Delta \setminus B(p)$ . Define for  $t \in \Delta$

$$(4. 9) \quad \mu_t(z) := \begin{cases} t\mu(z)/h, & z \in B(p), \\ Re(t)\mu(z)/h, & z \in E. \end{cases}$$

and the map

$$\begin{aligned} \Psi_E : \Delta &\hookrightarrow AT(\Delta), \\ t &\longmapsto [[\mu_t]]. \end{aligned}$$

We claim that  $\Psi_E$  is an isometric embedding. It suffices to check the equality:

$$(4. 10) \quad d_{AT}([\mu_t], [\mu_s]) = d_H(t, s), \quad t, s \in \Delta.$$

Using the previous notation, we have

$$(4.11) \quad \nu_{s,t} \circ f_t(z) = \begin{cases} \frac{1}{\tau} \frac{s-t}{1-st|\mu(z)|^2/h^2} \frac{\mu(z)}{h}, & z \in B(p), \\ \frac{1}{\tau} \frac{Re(s)-Re(t)}{1-Re(s)Re(t)|\mu(z)|^2/h^2} \frac{\mu(z)}{h}, & z \in E. \end{cases}$$

Firstly, it derives readily that restricted on  $f_t(B(p))$ ,

$$(4.12) \quad \|\nu_{s,t}\|_\infty = \left| \frac{s-t}{1-st} \right|.$$

Secondly, Lemma 4.4 indicates that restricted on  $f_t(E)$ ,

$$\|\nu_{s,t}\|_\infty \leq \left| \frac{s-t}{1-st} \right|.$$

By a similar argument to Case 1, we can get

$$h([\nu_{s,t}]) = k_0([\nu_{s,t}]) = \left| \frac{s-t}{1-st} \right|.$$

(4.10) follows immediately.

It remains to show that when  $r$  varies in a suitable range, equivalently, when  $E$  varies, the geodesic disks  $\Psi_E(\Delta)$  are mutually different. Fix  $t = \lambda i$  where  $\lambda \in (0, 1)$ . Since  $Re(t) = 0$ ,  $\mu_t(z) = 0$  on  $E$ . Therefore, none of inner points in the arc  $\partial\Delta \cap \partial E$  is a substantial boundary points for  $[[\mu_t]]$  but all points in the arc  $\partial\Delta \cap \partial B(p)$  are substantial boundary points. Therefore, when  $r$  varies,  $[[\mu_t]]$  are mutually different. Thus, we get infinitely many geodesic disks as required. Actually, in such a case, these geodesic disks contain the straight line  $\{[t\mu/h] : t \in (-1, 1)\}$ .

The proof of two cases above gives the following corollaries respectively.

**Corollary 2.** *Suppose  $[[\mu]]$  is not a substantial point in  $AT(\Delta)$ . Then there are infinitely many geodesics connecting  $[[\mu]]$  to the basepoint  $[[0]]$ .*

**Corollary 3.** *Suppose that  $[[\mu]]$  ( $\neq [[0]]$ ) is a substantial point in  $AT(\Delta)$  and  $\mu$  is a non-Strebel extremal representative. Then there are infinitely many geodesic disks containing the straight line  $\{[t\mu/k] : t \in (-1, 1)\}$ , where  $k = \|\mu\|_\infty \in (0, 1)$ .*

## 5. Geodesic planes in the tangent space

A geodesic plane in  $AZ(S)$  is the image of a map  $\Phi : \mathbb{C} \rightarrow AZ(S)$  which is an isometry with respect to the Euclidean metric on  $\mathbb{C}$  and the infinitesimal asymptotic metric on  $AZ(S)$ , respectively.

The infinitesimal version of Theorem 1 is as follows.

**Theorem 3.** *In the tangent space  $AZ(\Delta)$ , there are always infinitely many geodesic planes containing two points.*

*Proof.* It suffices to prove that for any  $[[\mu]]_{AZ} (\neq [[0]]_{AZ})$  in  $AZ(\Delta)$ , there are infinitely many geodesic planes passing through  $[[\mu]]_{AZ}$  and  $[[0]]_{AZ}$ . By Lemma 4.1, we can choose a non-Strebel extremal representative in  $[[\mu]]_{AZ}$ , say  $\mu$ . Since  $AZ(\Delta)$  is a Banach space, without loss of generality it is convenient to assume that  $\|\mu\|_\infty = b^*(\mu) = b([[ \mu ]])_{AZ} = 1$ .

*Case 1.*  $[[\mu]]_{AZ}$  is not an infinitesimal substantial point.

Suppose  $q \in \partial\Delta$  is not a substantial boundary point for  $[[\mu]]_{AZ}$ . By Lemma 4.2, we may assume  $b_q^*(\mu) < 1$  in addition.

By the definition of boundary dilatation, we can find a small neighborhood  $B(q)$  of  $q$  in  $\Delta$  such that  $|\mu(z)| \leq \rho < 1$  for some  $\rho > 0$  in  $B(q)$  almost everywhere. Therefore for any  $\zeta \in \partial\Delta \cap \partial B(q)$ ,  $b_\zeta^*(\mu) \leq \rho$ .

Choose  $\delta(z) \in M(\Delta)$  such that  $\|\delta\|_\infty \leq \beta < 1 - \rho$  and  $\delta(z) = 0$  when  $z \in \Delta \setminus B(q)$ .

Let  $\Sigma'$  be the collection of the complex-valued functions  $\sigma$  defined on  $\mathbb{C}$  with the following conditions:

- (A)  $\sigma$  is continuous with  $\sigma(0) = 0$  and  $\sigma(1) = 0$ ,
- (B)  $|s - t|\rho + |\sigma(t) - \sigma(s)|\beta \leq |s - t|$ ,  $t, s \in \mathbb{C}$ ,  $z \in B(q)$ .

Since  $\rho < 1$  and  $\beta < 1 - \rho$ ,  $\Sigma'$  contains uncountably many elements. In fact, if  $\sigma$  is a Lipschitz continuous function on  $\mathbb{C}$  with the following conditions,

- (i) for some small  $\alpha > 0$ ,  $|\sigma(s) - \sigma(t)| < \alpha|s - t|$ ,  $t, s \in \mathbb{C}$ ,
- (ii)  $\sigma(0) = 0$  and  $\sigma(1) = 0$ ,
- (iii)  $\rho + \alpha\beta < 1$ ,

then  $\sigma \in \Sigma'$ .

Given  $\sigma \in \Sigma'$ , define for  $t \in \mathbb{C}$ ,

$$(5.1) \quad \mu_t(z) = \begin{cases} t\mu(z), & z \in \Delta \setminus B(q), \\ t\mu(z) + \sigma(t)\delta(z), & z \in B(q), \end{cases}$$

and the map

$$\begin{aligned} \Phi_\sigma : \mathbb{C} &\hookrightarrow AZ(\Delta), \\ t &\longmapsto [[\mu_t]]_{AZ}. \end{aligned}$$

It is obvious that  $\Phi_\sigma(\mathbb{C})$  contains  $[[\mu]]_{AZ}$  and the basepoint  $[[0]]_{AZ}$ . We show that  $\Phi_\sigma$  is an isometric embedding. It is sufficient to verify that

$$(5.2) \quad \|[[\mu_s - \mu_t]]_{AZ}\| = |s - t|, \quad t, s \in \mathbb{C}.$$

At first, it is obvious that

$$\|\mu_s - \mu_t\|_\infty = |s - t|,$$

Suppose  $p \in \partial\Delta$  is a substantial boundary point for  $[[\mu]]_{AZ}$ . By Lemma 2.1 there is a degenerating Hamilton sequence  $\{\psi_n\} \subset Q^1(\Delta)$  towards  $p$  such that

$$1 = \lim_{n \rightarrow \infty} \operatorname{Re} \iint_{\Delta} \mu(z) \psi_n(z) dx dy.$$



Therefore, we have

$$|s - t| = \lim_{n \rightarrow \infty} \operatorname{Re} \iint_{\Delta} [\mu_s(z) - \mu_t(z)] e^{-i \arg(s-t)} \psi_n(z) dx dy, \quad s \neq t,$$

which implies the equality (5. 2).

It remains to show that there are infinitely many geodesic planes passing through  $[[\mu]]_{AZ}$  and  $[[0]]_{AZ}$  when  $\sigma$  varies over  $\Sigma'$  and  $\delta(z)$  varies over  $M(\Delta)$  suitably, respectively.

Choose  $\delta(z)$  in  $M(\Delta)$  such that (4. 8) holds. Fix a small  $t_0$  in  $(0, 1)$ . Choose  $\sigma \in \Sigma'$  such that  $\sigma(t) \equiv 0$  whenever  $|t| \geq t_0$  and  $\sigma(t) = \alpha t$  when  $t \in [0, t_0/2]$  where  $\alpha > 0$  satisfying  $\rho + \alpha\beta < 1$ . Note that when  $t \in [0, t_0/2]$ ,

$$\mu_t(z) = \begin{cases} t\mu(z)/h, & z \in \Delta \setminus B(q), \\ t\mu(z)/h + t\alpha\delta(z), & z \in B(q). \end{cases}$$

The geodesic planes  $\Phi_\sigma$  contain the geodesics  $G_\alpha = \{[[\mu_t]] : t \in [0, 1]\}$  respectively.

Due to the equality (4. 8), the geodesics  $G_\alpha$  are mutually different when  $\alpha$  varies in a small range. Therefore, the geodesic planes  $\Phi_\sigma(\mathbb{C})$  are mutually different.

If fix small  $\alpha > 0$  and let  $\delta$  vary suitably in  $M(\Delta)$ , then we can also get infinitely many geodesic planes as required.

*Case 2.*  $[[\mu]]_{AZ}$  is a substantial point.

Fix a boundary point  $p \in \partial\Delta$ . Let  $B(p) = \{z \in \Delta : |z - p| < r\}$  for small  $r > 0$  and  $E = \Delta \setminus B(p)$ . Define for  $t \in \mathbb{C}$

$$(5. 3) \quad \mu_t(z) := \begin{cases} t\mu(z), & z \in B(p), \\ \operatorname{Re}(t)\mu(z), & z \in E. \end{cases}$$

and the map

$$\begin{aligned} \Phi_E : \mathbb{C} &\hookrightarrow AZ(\Delta), \\ t &\longmapsto [[\mu_t]]_{AZ}. \end{aligned}$$

We claim that  $\Phi_E$  is an isometric embedding.

Note that

$$(5. 4) \quad \mu_s - \mu_t = \begin{cases} (s - t)\mu(z), & z \in B(p), \\ (\operatorname{Re}(s) - \operatorname{Re}(t))\mu(z), & z \in E. \end{cases}$$

It is easy to check the equality:

$$(5. 5) \quad \|[[\mu_s - \mu_t]]_{AZ}\| = |s - t|, \quad t, s \in \mathbb{C}.$$

It remains to show that when  $r$  varies in a suitable range, the geodesic planes  $\Phi_E(\mathbb{C})$  are mutually different. Fix  $t = \lambda i$  where  $\lambda \in (0, 1)$ . Since  $\operatorname{Re}(t) = 0$ ,  $\mu_t(z) = 0$  on  $E$ . Therefore, none of inner points in the arc  $\partial\Delta \cap \partial E$  is a substantial boundary points for  $[[\mu_t]]_{AZ}$  but all points in the arc  $\partial\Delta \cap \partial B(p)$  are substantial boundary points. Therefore, when  $r$  varies,  $[[\mu_t]]_{AZ}$  are mutually different. Thus, we get infinitely many geodesic planes as required. In particular, these geodesic planes contain the straight line  $\{[[t\mu]]_{AZ} : t \in \mathbb{R}\}$ .  $\square$

The following two corollaries follow from the proof of two cases above separately.

**Corollary 4.** *Suppose  $[[\mu]]_{AZ}$  is not a substantial point in  $AZ(\Delta)$ . Then there are infinitely many geodesics connecting  $[[\mu]]_{AZ}$  to the basepoint  $[[0]]_{AZ}$ .*

**Corollary 5.** *Suppose that  $[[\mu]]_{AZ} (\neq [[0]]_{AZ})$  is a substantial point in  $AZ(\Delta)$  and  $\mu$  is a non-Strebel extremal representative. Then there are infinitely many geodesic planes containing the straight line  $\{[[t\mu/k]]_{AZ} : t \in \mathbb{R}\}$ , where  $k = \|\mu\|_\infty > 0$ .*

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